Spatially-driven Instability in Lane-changing Models

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1 INTRODUCTION

Our scope is macroscopic models of n-lane highways that generalise the kinematic wave equation (Lighthill & Whitham, 1955) in the form

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial q_i}{\partial x} = F_{i-1} - F_i, \tag{1}$$

where ρ_i and q_i are respectively the density and flow in lane $i = 1, 2, \ldots, n$, and F_i is the net rate at which traffic in lane i moves to lane $i+1$. We take $F_i := F_i(\rho_i, \rho_{i+1})$ for $i = 1, 2, \ldots, n-1$, so that the rate of lane-changing between adjacent lanes is a function of their respective densities, and $F_0 = F_n = 0$ to model the left and right boundaries of the highway.

The simplest case is $n = 2$, where Munjal & Pipes (1971) chose $F_1 = k(\rho_2 - \rho_1)$ generalised to $F_1 = k(\rho_2 - \lambda\rho_1)$ by Holland & Woods (1997), who demonstrated how this apparently wave-like system can display a diffusive effect known as *Taylor dispersion*, that is well known in the fluid mechanics literature.

In the mathematical biology literature, there is an effect known as the *Turing Instability* (Turing, 1952) (or *Diffusion Driven Instability*), in which reaction-diffusion models can display spatiotemporal patterns, even when their spatially-independent dynamics (analogous to the RHS source terms in (1)) are stable. Since (1) can display diffusion-like properties, our goal here is to study whether spatiotemporal instabilities can be driven by lane-changing effects.

2 LINEAR STABILITY AND THE DISPERSION RELATION

We consider the linear stability of a spatially-independent equilibrium (known as 'uniform flows') $\boldsymbol{\rho}^* = (\rho_1^*, \rho_2^*, \ldots, \rho_n^*)^{\mathrm{T}}$, for which the lane-changing terms cancel out, so that $F_i(\rho_i^*, \rho_{i+1}^*) = F_{i-1}(\rho_{i-1}^*, \rho_i^*)$ for $i = 1, 2, \ldots, n$. Following e.g. Wilson (2008), we examine linear stability by the substitution $\boldsymbol{\rho} = \boldsymbol{\rho}^* + \tilde{\boldsymbol{\rho}}(x, t)$, where $\tilde{\boldsymbol{\rho}}$ is small. We try the ansatz $\tilde{\boldsymbol{\rho}}(x, t) = \mathrm{Re}(\mathbf{c}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{\lambda t})$, which for non-zero solutions $\mathbf{c}$ yields

$$|\mathbf{M} - \mathrm{i}k\mathbf{Q} - \lambda\mathbf{I}| = 0, \tag{2}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{Q}$ is the diagonal matrix $\mathrm{diag}(q_1'(\rho_1^*), q_2'(\rho_2^*), \ldots, q_n'(\rho_n^*)$, and $\mathbf{M}$ is a tridiagonal matrix with non-zero entries $m_{i,i-1} = (\mathrm{D}_1 F_{i-1})(\rho_{i-1}^*, \rho_i^*)$, $m_{i,i} = (\mathrm{D}_2 F_{i-1})$

$(\rho_{i-1}^*, \rho_i^*) - (\mathrm{D}_1 F_i)(\rho_i^*, \rho_{i+1}^*)$, and $m_{i,i+1} = -(\mathrm{D}_2 F_i)(\rho_i^*, \rho_{i+1}^*)$, where $(\mathrm{D}F)$ terms denote partial derivatives evaluated at the equilibrium. The determinant equation (2) can be used to derive the dispersion relation, which gives growth rates $\mathrm{Re}(\lambda)$ as a function of the wavenumber k.

Motivated by our search for Turing-like instabilities, we assume that the lane-changing source terms are 'ODE stable' — i.e., that the eigenvalues of $\mathbf{M}$ have negative (or zero) real parts — corresponding to $k = 0$ in (2). The question is then whether there is non-zero k for which $\mathrm{Re}(\lambda(k)) > 0$, which would indicate a spatially driven instability.

The simplest case for illustration is $n = 2$, where there is a single source term F_1. We then find

$$\mathbf{M} = \begin{pmatrix} -\mathrm{D}_1 F_1 & -\mathrm{D}_2 F_1 \\ +\mathrm{D}_1 F_1 & +\mathrm{D}_2 F_1 \end{pmatrix} \tag{3}$$

which has eigenvalues 0 and $\mathrm{D}_2 F_1 - \mathrm{D}_1 F_1$. The 0 eigenvalue has eigenvector $(\mathrm{D}_2 F_1, -\mathrm{D}_1 F_1)^{\mathrm{T}}$ best explained by reference to the Munjal and Pipes model, where it becomes $(1, 1)^{\mathrm{T}}$. This null-vector thus corresponds to translation along the continuous curve of equilibria $\rho_1 = \rho_2$. More broadly, for $n = 2$ systems, we expect the equilibria to trace out curves in the (ρ_1, ρ_2) plane. Generally speaking, we might expect that if the density of one lane were to increase, then for equilibrium, the density of the other lane should also increase, which implies that $\mathrm{D}_1 F_1$ and $\mathrm{D}_2 F_1$ must have opposite signs. If the equilibria are also ODE stable, it follows that

$$(\mathrm{D}_2 F_1) < 0 < (\mathrm{D}_1 F_1). \tag{4}$$

Unfortunately, it can be shown by analysing the quadratic

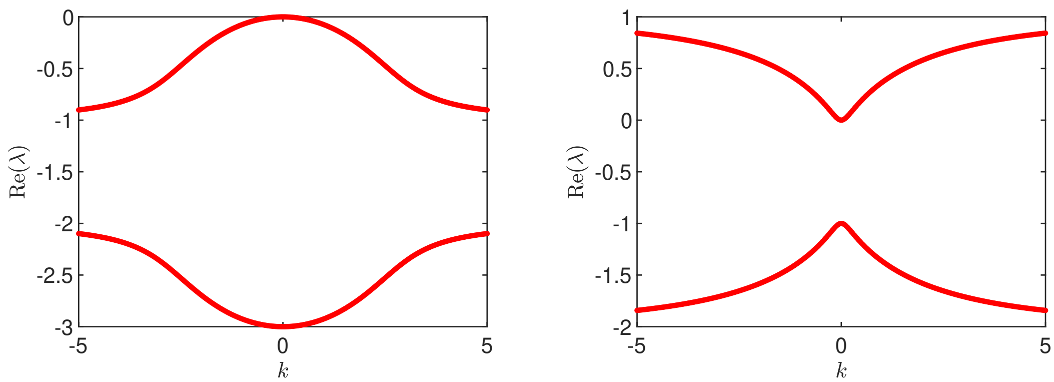
$$\lambda^2 + (A + B\mathrm{i})\lambda + C + D\mathrm{i} = 0 \tag{5}$$

where

$$A = \mathrm{D}_1 F_1 - \mathrm{D}_2 F_1, \quad B = k(q_1'(\rho_1^*) + q_2'(\rho_2^*)) \tag{6}$$

$$C = -k^2 q_1'(\rho_1^*) q_2'(\rho_2^*), \quad D = k(q_2'(\rho_2^*)\mathrm{D}_1 F_1 - q_1'(\rho_1^*)\mathrm{D}_2 F_1) \tag{7}$$

that arises from (2) that condition (4) implies that there is no spatial instability, see Figure 1. More complicated analyses for $n > 2$ suggest that spatial instability is only possible if the equilibrium relationships between lane densities ρ_i have negative slope — which does not seem realistic.



(a) $\mathrm{D}_1 F_1 = 1 \quad \mathrm{D}_2 F_1 = -2$

(b) $\mathrm{D}_1 F_1 = -1 \quad \mathrm{D}_2 F_1 = -2$

Figure 1: *Examples of the dispersion relation. For illustration, we set $q_1'(\rho_1^*) = 1$ and $q_2'(\rho_2^*) = 2$ for both examples. (a) Stable example (branches take negative values). (b) Unstable example (one branch is positive).*

3 CONTRIBUTION-BASED LANE-CHANGING MODELS

We propose a new approach, first introduced by Noble (2019), to macroscopic lane-changing models which extends beyond the state-of-the-art in two ways:

- Similar to Subraveti *et al.* (2019), it explicitly considers the *contributions* of flows of vehicles from lane i to $i+1$ *and* from $i+1$ to i, and not just their net outcome, by writing $F_i = f_{i,i+1} - f_{i+1,i}$.
- It explicitly distinguishes between the rate of lane-changing, interpreted at the level of a typical individual vehicle, from the size of the population that the rate applies to.

A natural choice for the contribution functions is $f_{i,j} = \rho_i g(v_j - v_i) h(\rho_j)$, describing a lane-changing rate from lane i to lane j which depends on the velocity advantage $v_j - v_i$ of lane j, and is multiplied by ρ_i (representing the population that the rate applies to). The function $h(\rho_j)$ should be decreasing and models how lane-changing into a dense lane becomes increasingly more difficult due to lack of space: henceforth, the choice $h(\rho) = \rho_{\max} - \rho$ suffices for our purposes where $\rho_{\max}$ is the jam density.

In the ordinary way, we might expect the function g to be increasing, to represent an increased incentive for changing to a lane that is much faster than our current one. However, we argue that in fact, a large speed difference cuts off lane changing altogether, since gap acceptance makes it impossible. Note that the domain of function g is $[-v_{\max}, v_{\max}]$ where $v_{\max}$ is the maximum velocity. In principle, the g functions could also be chosen differently to model asymmetry in the lane-changing process.

4 COMPUTATIONAL EXAMPLE OF INSTABILITY

In the simplest $n = 2$ case, we note that $(\mathrm{D}_1 F_1 + \mathrm{D}_2 F_1)|_{\rho_1=\rho_2} = 0$. This makes $\mathrm{D}_1 F_1$ and $\mathrm{D}_2 F_1$ have opposite signs, which implies by (4) that the natural equilibrium $\rho_1 = \rho_2$ does not have spatially driven instabilities. However, it can be shown, using asymptotic analysis, that for a given velocity-density relation, some contribution functions give rise to extra equilibria where the lane densities are very different and the lanes are almost decoupled. See Figure 2. In these situations, we can exhibit spatially driven instability. In the TRISTAN talk we will discuss what this implies for the solutions of the associated initial value problems.

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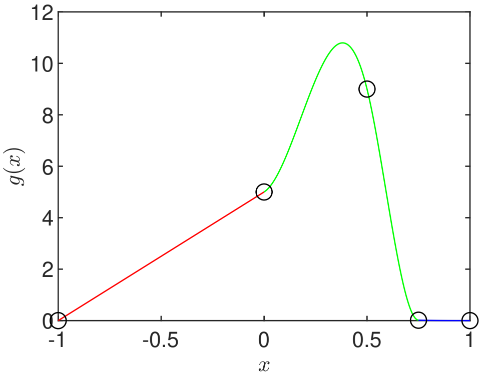
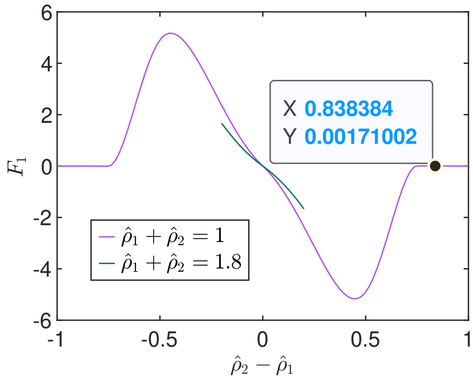
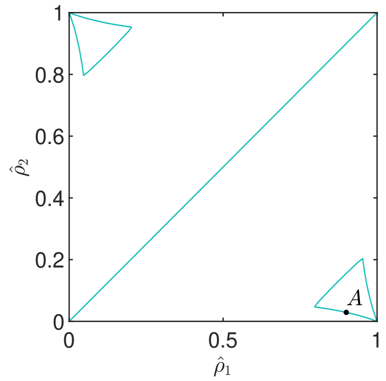


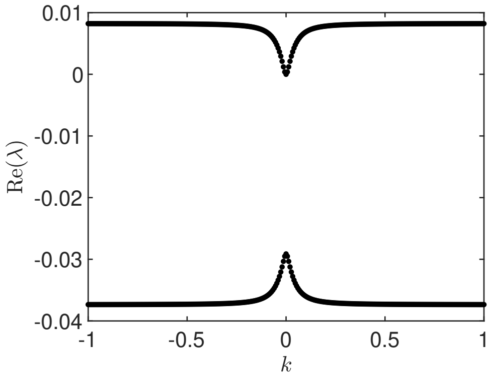
(a) Contribution function.



(b) Net rate of lane-changing.



(c) Branches of equilibria.



(d) Dispersion relation of point A.

Figure 2: *Example of spatially driven instability. (a) Contribution function that models a cut-off in lane-changing when the speed difference is too high. Our construction uses linear (red) and cubic (green / blue) splines. (b) Plot of* F_1 *when* $\hat{\rho}_1 + \hat{\rho}_2$ *is fixed, which suggests extra equilibria exist with* $\hat{\rho}_1 \neq \hat{\rho}_2$*. Here we use non-dimensional densities* $\hat{\rho}_1 := \rho_1/\rho_{\max}$ *and* $\hat{\rho}_2 := \rho_2/\rho_{\max}$*. (c) Plot of equilibria in the* $(\hat{\rho}_1, \hat{\rho}_2)$ *plane. (d) Dispersion relation of equilibrium point* A *(at which the equilibrium relationship slopes downwards). One solution branch is positive, indicating spatially driven instability.*