Multi-modal Transit Design with Stochastic Demands

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1 INTRODUCTION

Transit line-planning is a long standing transportation problem that is getting renewed attention due to the recent interest in multi-modal transit systems that incorporate Mobility-on-Demand (MoD) services. A multi-modal system aims to provide an efficient and cost effective mix of transportation modes to overcome the limitations of any single mode. One of the major challenges and opportunities in designing such systems is the fact that part of the system is fixed (e.g., bus routes are fixed), while the demand responsive MoD component can dynamically adjust to demand uncertainty. There have been a number of recent algorithms proposed to design and operate such systems Maheo *et al.* (2019), Pinto *et al.* (2020), Dalmeijer & Van Hentenryck (2020), Périvier *et al.* (2021), Banerjee *et al.* (2021)). However, most of these studies assume that the underlying travel demand is deterministic, even though the demand is stochastic—and importantly, a factor that makes multi-modal transit systems with MoD components more attractive.

This work aims to provide some guidance on how demand uncertainty should be accounted for with respect to a general class multi-modal transit network planning problems, by providing theoretical bounds on common approaches used in the literature (without theoretical justification). In particular, we show that when the demand for each origin-to-destination pair follows a bounded distribution independent of time and the known information about the distribution is its expectation and its upper and lower bounds, we can replace the actual (stochastic) demand with its expectation to approximate the (stochastic) true values as long as the span of planning time length is reasonably long. We also show that this framework can be used to compare between two different combinations of modes.

2 Modeling

Although our framework is generally applicable to many models of transit network design, for clarity and conciseness we illustrate it with one such model in this abstract, specifically a derivative of the ODMTS (On-Demand Multimodal Transit Systems) model from Dalmeijer & Van Hentenryck (2020). Let $G = (V, A)$ with vertices $V = \{1, \dots, n\}$ and arc set A. Let M be the set of possible transportation modes, and F be set of the frequency options for transportation modes. Then $a \in$ A is defined by $a = (i, j, f, m) \in V \times V \times M \times F$, $\forall i \neq j$.

To design a multi-modal transit network over T time intervals, we need to determine which arcs are going to be opened to serve passengers over the given time span. Let the decision variables be $z := (z_a)_{a\in A}$ such that $z_a = 1$ if a is opened and $z_a = 0$ otherwise. Let $\beta_a \geq 0$ be the operating cost for each time interval and $a \in A$, and c_a be the travelling cost incurred by one passenger using arc a within one time interval.

Once z is fixed, a passenger's trip r is routed based on the given network design, her origin node, destination node and specified routing rules for all the passengers. For example, in the ODMTS model, one of the rules is that each passenger can use at most K arcs in G, where K is a fixed positive integer. Then let $\mathcal{R}$ be the set of all passenger trips, and let each trip $r \in \mathcal{R}$ be defined by $(o(r), d(r)) \in V \times V$, where $o(r)$ is the origin of r, $d(r)$ is a destination of r. Note that here we assume the route for r is a simple path, i.e. each arc a is only traveled once by r. Also, let $p_t(r)$ be the number of passengers on r within time interval t.

Given route planning rules and a network design z, there is a fixed route and travelling cost $RC(z, r)$ incurred by a passenger of trip r traveling within one time interval. Note that here we implicitly state that for passengers of the same trip, the corresponding travelling costs are equal. This is because we assume that each transportation mode has no capacity constraints and the routing rules are applicable to all the passengers. Now we can formulate the whole problem as the following.

$$
\begin{aligned}
\min_z \quad & \sum_{t=1}^{T}\left(\left(\sum_{a\in A}\beta_a z_a\right) + \sum_r p_t(r) RC(z,r)\right), && && \text{(1a)}\\
\text{s.t.} \quad & \sum_{a\in\delta^+(i,m)f(a)z_a} f(a)z_a - \sum_{a\in\delta^-(i,m)f(a)z_a} f(a)z_a = 0 && \forall i \in V, m\in M && \text{(1b)}\\
& \sum_{f\in F,(i,j,m,f)\in A} z_{(i,j,m,f)} \leq 1 && \forall i \in V, m\in M, && \text{(1c)}\\
& z_a \in \{0,1\} && \forall a \in A, && \text{(1d)}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\min_z \quad & \sum_{a\in A}\beta_a z_a + \sum_r \frac{\sum_{t=1}^{T} p_t(r)}{T} RC(z,r), && && \text{(2a)}\\
\text{s.t.} \quad & \sum_{a\in\delta^+(i,m)f(a)z_a} f(a)z_a - \sum_{a\in\delta^-(i,m)f(a)z_a} f(a)z_a = 0 && \forall i \in V, m\in M && \text{(2b)}\\
& \sum_{f\in F,(i,j,m,f)\in A} z_{(i,j,m,f)} \leq 1 && \forall i \in V, m\in M && \text{(2c)}\\
& z_a \in \{0,1\} && \forall a \in A. && \text{(2d)}
\end{aligned}
$$

Here (2b) guarantees the in-flow is equal to the out-flow for each mode and each node, and (2c) assures each arc can only adopt one frequency.

Note that the formulation here is different from the one by Dalmeijer & Van Hentenryck (2020) in two aspects: (1) we add time t as a new dimension to our model, (2) routing rules can be more general here than in ODMTS model in which the number of transfers is restricted, and trips are routed for minimal costs.

When $p_t(r)$ is known for every t and r, we can solve (2) with an exact algorithm or heuristic. However, in real life, $p_t(r)$ is usually uncertain (especially when T is large). As we have mentioned, in this work we try to deal with the issue caused by the uncertainty. Specifically, we will assume that $p_t(r) \in [l_r, u_r]$ for $u_r \geq l_r \geq 0$, and $\mathbb{E}(p_t(r))$ is known, and denoted by E_r. Without loss of generality, we can assume $p_t(r) \in [0, u]$ by setting $u = \max_r u_r$. Note that $p_t(r)$ is independent w.r.t. t but not necessarily w.r.t. r. We also assume that there exists $\lambda > 1$ such that for each r, $\lambda E_r \geq u_r$.

3 Methodology and Results

We will show that under the above assumptions, when T is larger than a threshold polynomial in the input size, the solution of (2) can be approximated well by the following deterministic optimization problem.

$$\begin{aligned}
\min_{z} \quad & \sum_{a\in A} \beta_a z_a + \sum_{r} E_r \cdot RC(z,r), && && \text{(3a)}\\
\text{s.t.} \quad & \sum_{a\in\delta^{+}(i,m)f(a)z_a} f(a)z_a - \sum_{a\in\delta^{-}(i,m)f(a)z_a} f(a)z_a = 0 && \forall i\in V, m\in M, && \text{(3b)}\\
& \sum_{f\in F,(i,j,m,f)\in A} z_{(i,j,m,f)} \le 1 && \forall i\in V, m\in M, && \text{(3c)}\\
& z_a \in \{0,1\} && \forall a\in A, && \text{(3d)}
\end{aligned}$$

where E_r and $RC(z,r)$ are defined same as in Section 2.
For simplicity, let $\mathrm{obj}_T(z) := \sum_{a\in A}\beta_a z_a + \sum_r \frac{\sum_{t=1}^{T} p_t(r)}{T} RC(z,r)$, and $|\mathcal{R}|$ be the size of the trip set $\mathcal{R}$.

Our first result shows that as T increases, the solution of (3) converges[1] to an optimal solution of (2).

Theorem 3.1. Let OPT_T and OPT_E be the optimal values of (2), and (3). Also, let $z^{(T)} := (z_a^{(T)})_{a\in A}$ and $z^{(E)} := (z_a^{(E)})_{a\in A}$ be the optimal solution to (2) and (3). Then we have

$$\begin{aligned}
&\mathrm{OPT}_T \xrightarrow{a.s.} \mathrm{OPT}_E && \text{as } T\to\infty, && (4)\\
&\mathrm{obj}_T(z^{(E)}) - \mathrm{OPT}_T \xrightarrow{a.s.} 0 && \text{as } T\to\infty. && (5)
\end{aligned}$$

Although the above result shows that the estimation converges to the real optimal value as T increases, it will be unrealistic if T needs to be exponential in the input size of the original problem to attain a desired error rate. Fortunately, the following results show that T only needs to be polynomial in the input size so that the approximation is close to the original optimal solution.

Theorem 3.2. With the same notation as in Theorem 3.1, for $0<\alpha\le 1$, we have

$$\mathbb{P}\left\{\left|\mathrm{obj}_T(z^{(E)}) - \mathrm{OPT}_T\right| \ge \alpha\mathrm{OPT}_T\right\} \le 2|\mathcal{R}|\exp\left(-\frac{T\alpha^2}{(2(1+\alpha)|\mathcal{R}|+\alpha)^2\lambda^2}\right). \tag{6}$$

Hence for $\alpha > 1$, we have

$$\mathbb{P}\left\{\left|\mathrm{obj}_T(z^{(E)}) - \mathrm{OPT}_T\right| \ge \alpha\mathrm{OPT}_T\right\} \le 2|\mathcal{R}|\exp\left(-\frac{T}{(4|\mathcal{R}|+1)^2\lambda^2}\right). \tag{7}$$

Corollary 3.3. Given $0<\delta<1$ and $0<\alpha\le 1$, if $T \ge \log(\frac{2|\mathcal{R}|}{\delta})\lambda^2\left(2(1+\frac{1}{\alpha})\mathcal{R}+1\right)^2$, we have

$$\mathbb{P}\left\{\left|\mathrm{obj}_T(z^{(E)}) - \mathrm{OPT}_T\right| \ge \alpha\mathrm{OPT}_T\right\} \le \delta. \tag{8}$$

[1]Here $\xrightarrow{a.s.}$ refers to almost surely convergence.

In some application scenarios, comparison between different combinations of modes is needed. For example, a transit agency might be deciding whether to introduce a BRT or light-rail service in combination with regular bus service and a demand responsive shuttle service. To estimate the cost ratio, we need to estimate the corresponding costs by solving (3) with different transportation modes constraints, and show that the estimated ratio of costs is close to the true ratio if the estimated solutions are employed. Particularly, we prove the following theorem.

Theorem 3.4. Given $A_1, A_2 \subseteq A$, let OPT_1 and OPT_2 be the solutions to (3) with added constraints $z_a = 0$ for $a \in A_1$ and $z_a = 0$ for $a \in A_2$ respectively. Let $z^{(1)}$ and $z^{(2)}$ be the corresponding solutions. Then for $\alpha > 0$, we have

$$\mathbb{P}\left\{\left|\frac{\mathrm{obj}_T(z^{(2)})}{\mathrm{obj}_T(z^{(1)})} - \frac{\mathrm{OPT}_2}{\mathrm{OPT}_1}\right| \geq \alpha \frac{\mathrm{OPT}_2}{\mathrm{OPT}_1}\right\} \leq 4|\mathcal{R}| \exp\left(-\frac{\alpha^2 T}{(2+\alpha)^2 |\mathcal{R}|^2 \lambda^2}\right) \tag{9}$$

In Theorem 3.4, A_1 and A_2 are the sets of arcs that are not considered. Consider the case of comparing the solution $z^{(1)}$ corresponding to the case when only a subset of transportation modes $M^1 \subseteq M$ can be selected with the solution $z^{(2)}$ corresponding to only $M^2 \subseteq M$ being available for adoption. Then we can let A_1 be the set of arcs that correspond to the modes not in M^1, and A_2 be the set of arcs that correspond to the modes not in M^2.

In addition to this concentration result, we can also prove the convergence and a threshold for T similar to Theorem 3.1 and Corollary 3.3.

Corollary 3.5. With the same notation as in Theorem 3.4, we have

$$\frac{\mathrm{obj}_T(z^{(2)})}{\mathrm{obj}_T(z^{(1)})} \xrightarrow{a.s.} \frac{\mathrm{OPT}_2}{\mathrm{OPT}_1} \qquad \text{as } T \to \infty. \tag{10}$$

Corollary 3.6. Given $0 < \delta < 1$ and $0 < \alpha \leq 1$, if $T \geq \log(\frac{4|\mathcal{R}|}{\delta})\lambda^2 \left((1+\frac{2}{\alpha})|\mathcal{R}|\right)^2$, we have

$$\mathbb{P}\left\{\left|\frac{\mathrm{obj}_T(z^{(2)})}{\mathrm{obj}_T(z^{(1)})} - \frac{\mathrm{OPT}_2}{\mathrm{OPT}_1}\right| \geq \alpha \frac{\mathrm{OPT}_2}{\mathrm{OPT}_1}\right\} \leq \delta. \tag{11}$$

Remark. At this point, we have shown that the optimal solution to (2) is an estimation with desired error rate as long as T is polynomially large. However, solving (2) exactly can be time-consuming. Fortunately, if there is an algorithm or heuristic that returns a near-optimal solution to (3) with some approximation ratio, then it can be demonstrated that this solution is also a near-optimal solution to (2) with a similar performance guarantee as long as T is polynomial in the input size of the original problem. We will also conduct numerical experiments in the future.

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